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ABSTRACT

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This study contains a digest of Minkowski's theory of electrodynamics of moving media in the three-dimensional form and a critical review of some current writings on this subject from the point of view of Minkowski's theory. The invariant nature of the Maxwell-Minkowski equations is explained in terms of a conventional language. The important role played by the constitutive relations in formulating a complete theory of electrodynamics of moving media is pointed out.

Author

CONTENTS

A STUDY OF ELECTRODYNAMICS OF MOVING MEDIA

PREFACE

"... Minkowski, in 1908, at long last in full possession of the principle of relativity, was the first to solve the problem completely."

Arnold Johannes Wilhelm Sommerfeld (1868-1951)

INTRODUCTION

The correct formulation of the electrodynamics of moving media has been a challenge to scientists in the pre-relativistic era as well as in modern times. As we know from the history of the subject, the theory was formulated by Minkowski[1] three years after Einstein's enunciation of the special theory of relativity. Many physicists, specializing in relativistic theory, undoubtedly have a comprehensive knowledge on this subject. Unfortunately, from the author's own experience, Minkowski's great work has not penetrated very deeply into the minds of many people. As a result, many of us must learn the subject the hard way and encounter endless frustrations. Of the contemporary writings there is no doubt that Sommerfeld's masterful book[2] contains all the details of Minkowski's theory in modern form. That great teacher, perhaps accustomed to the group of brilliant students surrounding him during his entire life, did not elaborate some of the technical aspects of the theory where most of us need guidance. Cullwick,[3] in his scholarly book, attempted to present the subject, using conventional language. However, because of his "unorthodox" approach and his liberal attitude and strong leaning towards various possible "physical" models, not enough emphasis seems to have been placed upon the unique features of Minkowski's theory. The purposes of this article are, first, to give a digest of Minkowski's theory; and second, to give a critical review of some current writings on this subject from the point of view of Minkowski's theory. In particular, we shall point out the importance of the constitutive relations in providing a complete theory of electrodynamics of moving media.

Let us emphasize from the very beginning that without Einstein's special theory of relativity the electrodynamics of moving media would remain a mysterious problem within the framework of classical physics. While the mathematics of the special theory of relativity is not complicated, it is the physics, more specifically the concept of invariance, that has blocked our comprehension of the beauty of

Minkowski's theory. Since it is not the purpose of this article to give a full account of Minkowski's theory in its original form, we intend to restate the theory using a language familiar to those readers who already know the electrodynamics of a stationary medium. For that matter we shall first review Maxwell's equations for a stationary medium and point out the importance of the constitutive relations.

MAXWELL'S EQUATIONS IN THE INDEFINITE FORMS AND THE CONSTITUTIVE RELATIONS

As stated in most modern text books of electromagnetic theory, the standard form of Maxwell equations are given by

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{2}$$

They are supplemented by two auxiliary equations

$$\nabla \cdot \vec{D} = \rho \tag{3}$$

$$\nabla \cdot \vec{B} = 0 \tag{4}$$

The nomenclature is given by:

$\vec{E}$, $\vec{D}$ = electric field vectors

$\vec{H}$, $\vec{B}$ = magnetic field vectors

$\vec{J}$, ρ = free-current and free-charge densities .

When the constitutive relations between the four field vectors are unknown or unspecified Eqs. (1-4) are insufficient for us to permit a solution. We shall therefore designate Eqs. (1-2), as long as the constitutive relations are not specified, as the Maxwell equations in the "indefinite" form. For historical reasons, and perhaps many others, the constitutive relations have always been introduced in terms of two material field vectors $\vec{P}$ and $\vec{M}$, the polarization density and the magnetization density. These are usually defined, for stationary media, in the following two equations:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P} \tag{5}$$

$$\vec{B} = \mu_o (\vec{H} + \vec{M}) \; . \tag{6}$$

As we are going to elaborate later, the misunderstanding of the electrodynamics of moving media is partly due to the delicate concepts attached to these two quantities. Now, as long as the relations between $\vec{P}$, $\vec{M}$ and $\vec{E}$, $\vec{H}$ are still unknown or unspecified, the constitutive relations are not yet specified, and we have not changed the indefinite nature of Eqs. (1-2). However, the latter may now be presented in four apparently different forms. They are:

<u>Form I, EPBM</u>

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \tag{7}$$

(8) $$\nabla \times \left(\frac{\vec{B}}{\mu_o}\right) = \frac{\partial}{\partial t} (\epsilon_o \vec{E} + \vec{P}) + \nabla \times \vec{M} + \vec{J}$$

<u>Form II, EPHM</u>

(9) $$\nabla \times \vec{E} = - \frac{\partial}{\partial t} \mu_o(\vec{H} + \vec{M})$$

(10) $$\nabla \times \vec{H} = \frac{\partial}{\partial t} (\epsilon_o \vec{E} + \vec{P}) + \vec{J}$$

<u>Form III, DPBM</u>

(11) $$\nabla \times \left(\frac{\vec{D}}{\epsilon_o}\right) = - \frac{\partial \vec{B}}{\partial t} + \nabla \times \left(\frac{\vec{P}}{\epsilon_o}\right)$$

(12) $$\nabla \times \left(\frac{\vec{B}}{\mu_o}\right) = \frac{\partial \vec{D}}{\partial t} + \nabla \times \vec{M} + \vec{J}$$

<u>Form IV, DPHM</u>

(13) $$\nabla \times \left(\frac{\vec{D}}{\epsilon_o}\right) = - \frac{\partial}{\partial t} \mu_o(\vec{H} + \vec{M}) + \nabla \times \left(\frac{\vec{P}}{\epsilon_o}\right)$$

(14) $$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad .$$

Several schools of though have arisen in arguing as to which of these is the most desirable form. Form I, the so-called Amperian model, is favored by some people mainly because the terms $\partial\vec{P}/\partial t$ and $\nabla \times \vec{M}$ can be vividly interpreted as the electric polarization current and the electric magnetization current, respectively. The term $\partial(\mu_0\vec{M})/\partial t$ in Form II can also be vividly interpreted as a magnetic magnetization current. Forms III and IV played an insignificant role in the history of electrodynamics. Now, we would like to cast one definite opinion in regard to these forms. As long as Maxwell's equations are written in their indefinite form all these variants are equally acceptable. The reason for this strong statement is that once the constitutive relations are known or specified all of them reduce to one definite form, and it is this final form of Maxwell's equations with the constitutive relations that provide us a working model. In other words, it is meaningless to speak of the solutions of the Maxwell's equations in their indefinite forms. Thus, for a stationary, isotropic, homogeneous, linear medium, we have

$$\vec{D} = \epsilon \vec{E} \tag{15}$$

$$\vec{B} = \mu \vec{H} \tag{16}$$

where ϵ and μ denote, respectively, the permittivity and permeability of the medium. With these additional relations Eqs. (1-2) have only one definite form although we may still present them in terms of $\vec{E}$ and $\vec{H}$, $\vec{E}$ and $\vec{B}$, $\vec{D}$ and $\vec{H}$, or $\vec{D}$ and $\vec{B}$. The difference, however, is trivial. With this much discussion of the electrodynamics of stationary media we pass to the electrodynamics of moving media.

MAXWELL-MINKOWSKI EQUATIONS IN THE INDEFINITE FORM

For clarity, we shall arbitrarily divide Minkowski's theory of electrodynamics of moving media into two distinct parts. The first part deals with the invariant nature of Maxwell's equations. It is, perhaps, mainly a misunderstanding of this part of Minkowski's theory that has created many apparent paradoxes in the past and the present. Expressed in terms of our usual language, the first part of his theory shows that Maxwell's equations as stated in (1-2), originally formulated by Maxwell for stationary media, are also valid for a medium which is moving uniformly with a velocity $\vec{v}$ with respect to a fixed reference system, usually the laboratory system. To avoid any confusion we will rewrite Eqs. (1-4) with a subscript "m" attached to all the field quantities as well as the current-density and charge density. They are:

(17) $$\nabla \times \vec{E}_m = -\frac{\partial \vec{B}_m}{\partial t}$$

(18) $$\nabla \times \vec{H}_m = \frac{\partial \vec{D}_m}{\partial t} + \vec{J}_m$$

(19) $$\nabla \cdot \vec{D}_m = \rho_m$$

(20) $$\nabla \cdot \vec{B}_m = 0$$

where the electromagnetic quantities are functions of position and time defined in the laboratory system in the presence of the moving medium. From now on we shall refer to Eqs. (17-20) as the Maxwell-Minkowski equations.* Since the velocity vector $\vec{v}$ of the moving medium does not appear explicitly in Eqs. (17-20), it must be contained implicitly in the constitutive relations of the field vectors. The second part of Minkowski's theory is to determine the constitutive relations based upon the special theory of relativity, on the assumption that the constitutive relations for the medium at rest are known. But before we present the second part, we shall review various indefinite forms of the Maxwell-Minkowski's equations that the author is aware of.

Form A, EPBM[4]

This is the Amperian model which we encountered before in the section dealing with stationary media. Although, the form of the equations is not changed we shall attach a subscript "a" to designate all the quantities pertaining to this model for the moving media. They are

$$\nabla \times \vec{E}_a = - \frac{\partial \vec{B}_a}{\partial t} \tag{21}$$

* This designation is not in conflict with Sommerfeld's usage of the same term (see Reference 2, p. 280) although he uses it to designate the invariant forms of Maxwell's equations in two inertial systems (a relativistic language).

(22) $$\nabla \times \left(\frac{\vec{B}_a}{\mu_o}\right) = \frac{\partial(\epsilon_o \vec{E}_a + \vec{P}_a)}{\partial t} + \nabla \times \vec{M}_a + \vec{J}_a$$

(23) $$\nabla \cdot (\epsilon_o \vec{E}_a + \vec{P}_a) = \rho_a$$

(24) $$\nabla \cdot \vec{B}_a = 0$$

Form B, EHPMv

This form was first derived by Chu,[5] using a kinematic method. It can also be derived from the method of motional flux.[6*] They are:

(25) $$\nabla \times (\vec{E}_c + \mu_o \vec{M}_c \times \vec{v}) = -\frac{\partial \mu_o(\vec{H}_c + \vec{M}_c)}{\partial t}$$

(26) $$\nabla \times (\vec{H}_c - \vec{P}_c \times \vec{v}) = \frac{\partial(\epsilon_o \vec{E}_c + \vec{P}_c)}{\partial t} + \vec{J}_c$$

(27) $$\nabla \cdot (\epsilon_o \vec{E}_c + \vec{P}_c) = \rho_c$$

(28) $$\nabla \cdot \mu_o (\vec{H}_c + \vec{M}_c) = 0 \quad .$$

The subscript "c" is used to designate all quantities pertaining to this formulation.

In addition to the above two forms we may mention the work by Panofsky and Phillips[7] where the motional flux method is applied to

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* The method of motional flux when expressed in EBHD form is the same as the Lorentz-Pauli equations as described in Reference (2). It is merely a recasting of the invariant nature of Maxwell equations from the point of view of Minkowski's theory in the three dimensional form.

Faraday's law, and a kinematic method is applied to the Maxwell-Ampere law, in treating a non-magnetic moving medium. We would like to point out that the polarization current defined in that work has a quite different meaning than the polarization current defined by Chu.

As for the case of stationary media as long as the constitutive relations are unknown or unspecified all these forms of the Maxwell-Minkowski equations remain indefinite and are equally acceptable provided that they are self-consistent. In fact, we may formulate a more general principle of saying that <u>any non-singular linear transformation of the field vectors, $\vec{E}_m$, $\vec{B}_m$, $\vec{H}_m$, and $\vec{B}_m$ into another set of four field vectors is acceptable</u>. Of course, the transformed vectors may not have an easily identifiable physical significance. The inherent reason for offering such a general principle is that the invariant nature of the Maxwell-Minkowski equations is still preserved under such a transformation. This principle, of course, covers all the indefinite forms which we have mentioned for the electrodynamics of moving media as well as of stationary media. Based upon this principle, we see that Forms A and B are equally acceptable. The transformations that relate the field vectors in Forms A and B and the Maxwell-Minkowski field vectors are given below:

Form A

(29) $\vec{E}_a = \vec{E}_m$

(30) $\vec{B}_a = \vec{B}_m$

(31) $\frac{\vec{B}_a}{\mu_o} - \vec{M}_a = \vec{H}_m$

(32) $\epsilon_o \vec{E}_a + \vec{P}_a = \vec{D}_m$

(33) $\vec{J}_a = \vec{J}_m$

(34) $\rho_a = \rho_m$

Form B

(35) $\vec{E}_c + \mu_o \vec{M}_c \times \vec{v} = \vec{E}_m$

(36) $\mu_o (\vec{H}_c + \vec{M}_c) = \vec{B}_m$

(37) $\vec{H}_c - \vec{P}_c \times \vec{v} = \vec{H}_m$

(38) $\epsilon_o \vec{E}_c + \vec{P}_c = \vec{D}_m$

(39) $\vec{J}_c = \vec{J}_m$

(40) $\rho_c = \rho_m$.

These relations are essentially linear transformations of the type referred to above. The field vectors $\vec{E}_c$ and $\vec{H}_c$ obviously do not have the same meaning as the field vectors $\vec{E}_m$ and $\vec{H}_m$ defined in the

Maxwell-Minkowski equations. However, when $v = 0$ they are the same as $\vec{E}_m$ and $\vec{H}_m$. Having explained the invariant nature of the Maxwell-Minkowski equations we shall present the second part of Minkowski's theory, namely, the determination of the constitutive relations for a moving medium.

THE DEFINITE FORM OF MAXWELL-MINKOWSKI EQUATIONS

The problem can be stated as follows: Suppose that the constitutive relations of a stationary, isotropic, homogeneous, linear medium are

$$\vec{D} = \epsilon \vec{E} \tag{41}$$

$$\vec{B} = \mu \vec{H} \tag{42}$$

$$\vec{J} = \sigma \vec{E} \; . \tag{43}$$

If the medium is now moving with a uniform velocity $\vec{v}$ what would be the constitutive relations for the electrodynamic quantities $\vec{E}_m$, $\vec{B}_m$, $\vec{H}_m$, $\vec{D}_m$, $\vec{J}_m$, and ρ_m? An exact answer to this problem was one of the major contributions by Minkowski, based upon the special theory of relativity. The complete answer has been reproduced in many books of electromagnetic theory, perhaps, without too much explanation. From Sommerfeld's formulae,[8] if terms of the order of

$(v/c)^2$ are neglected, where v denotes the magnitude of the velocity of the moving medium, and c, the velocity of light, one obtains the following formulae:

$$\vec{D}_m = \epsilon\, \vec{E}_m + (\epsilon\mu - \epsilon_o \mu_o)\, \vec{v} \times \vec{H}_m \tag{44}$$

$$\vec{B}_m = \mu\, \vec{H}_m - (\epsilon\mu - \epsilon_o \mu_o)\, \vec{v} \times \vec{E}_m \tag{45}$$

$$\vec{J}_m = \rho_m \vec{v} + \sigma\, [\vec{E}_m + \vec{v} \times \vec{B}_m] \quad . \tag{46}$$

We have so far stayed away from the relativistic part of the theory except to use the results. As far as accuracy is concerned, we shall consider Eqs. (44-46) as correct, just as we consider Newtonian mechanics to be, as long as $v << c$. When $\vec{v} = 0$, these equations reduce to the well-known constitutive relations for a stationary medium. When $\mu\epsilon = \mu_o\epsilon_o$, $\sigma = 0$ and $\rho_m = 0$ they reduce to the constitutive relations for empty space. This is one of the beautiful aspects of Minkowski's theory. Once the constitutive relations are known there is one and only one definite form of the Maxwell-Minkowski equations and this is Minkowski's theory of electrodynamics of moving media.

Returning now to other formulations (A and B) one may, of course, start with these forms and determine the corresponding constitutive relations by means of relativistic theory.* In view of the relations expressed by (29-40) this is, of course, not necessa because we can give an interpretation of all these formulations from the point of view of Minkowski's theory.

Interpretation of the EBPM Formulation (Form A)

By making use of Eqs. (29-32) and (44-45) one can easily find that

(47) $$\vec{E}_a = \vec{E}_m$$

(48) $$\vec{B}_a = \mu \vec{H}_m - (\mu\epsilon - \mu_o\epsilon_o) \vec{v} \times \vec{E}_m$$

(49) $$\mu_o \vec{M}_a = (\mu - \mu_o) \vec{H}_m - (\mu\epsilon - \mu_o\epsilon_o) \vec{v} \times \vec{E}_m$$

(50) $$\vec{P}_a = (\epsilon - \epsilon_o) \vec{E}_m + (\mu\epsilon - \mu_o\epsilon_o) \vec{v} \times \vec{H}_m \quad .$$

The relations between $\vec{M}_a$, $\vec{P}_a$ and $\vec{B}_a$, $\vec{E}_a$ can readily be found if desired.

* In Boffi's dissertation, Reference (4), he formulated the problem relativistically. But, instead of following the track of Minkowski, he was more concerned about the energy relationship based upon the Amperian model. The constitutive relations were not discussed. Appendix I of Reference (5) gives a formal representation of Eqs. (25-28) in the four dimensional form. It does not have, at least in spirit, the same significance as Minkowski's four dimensional formulation that enables him to determine the constitutive relations as stated by Eqs. (44-46).

Interpretation of the EHPMv Formulation* (Form B)

The EHPMv formulation has attracted much attention recently. The steps that lead to this formulation based upon the kinematic method are very systematic. The theory however, is obscured by the absence of a thorough discussion of the constitutive relations between the field vectors defined in that formulation; hence it remains as one of the indefinite forms.

In the original work,[9] reference (5), the constitutive relations are stated without derivation. They appear to stem from physical intuition. We shall here verify these relations based upon Minkowski's theory. By making use of Eqs. (35-38) and (44-45), and neglecting terms of the order of $(v/c)^2$, that are justified in view of the same approximations as are involved in Eqs. (44-45), one finds that

$$\vec{E}_c = \vec{E}_m + (\mu - \mu_o)\,\vec{v} \times \vec{H}_m \tag{51}$$

$$\vec{H}_c = \vec{H}_m - (\epsilon - \epsilon_o)\,\vec{v} \times \vec{E}_m \tag{52}$$

$$\mu_o \vec{M}_c = (\mu - \mu_o)\,(\vec{H}_m - \epsilon\, \vec{v} \times \vec{E}_m) \tag{53}$$

$$\vec{P}_c = (\epsilon - \epsilon_o)\,(\vec{E}_m + \mu\, \vec{v} \times \vec{H}_m) \quad . \tag{54}$$

* Many of the fine details contained in this section result from a valuable discussion with Professor H. C. Ko.

Under the same approximation, it can be shown that

$$\mu_o \vec{M}_c = (\mu - \mu_o)(\vec{H}_c - \epsilon_o \vec{v} \times \vec{E}_c) \tag{55}$$

$$\vec{P}_c = (\epsilon - \epsilon_o)(\vec{E}_c + \mu_o \vec{v} \times \vec{H}_c) \ . \tag{56}$$

The last two equations correspond to the constitutive relations for the field vectors defined in the EHPMv-formulation, which are now derived from Minkowski's theory. By comparing Eqs. (53-54) with Eqs. (55-56) we see that

$$\vec{H}_m - \epsilon \vec{v} \times \vec{E}_m = \vec{H}_c - \epsilon_o \vec{v} \times \vec{E}_c \tag{57}$$

$$\vec{E}_m + \mu \vec{v} \times \vec{H}_m = \vec{E}_c + \mu_o \vec{v} \times \vec{H}_c \ . \tag{58}$$

The right-hand side of Eqs. (57-58) are the "effective" $\vec{H}$ and $\vec{E}$ fields as designated by the authors in the EHPMv formulation. As seen by Eqs. (57-58) these fields are closely analogous to similar quantities in the Minkowski's formulation. We have thus identified all the field vectors in the EHPMv formulation through Minkowski's theory. As we have emphasized before there is only one definite form of the Maxwell-Minkowski equations once the constitutive relations are revealed.

CONCLUSION

As a result of this study it is hoped that we have clarified certain misunderstandings about Minkowski's theory, so that it can be appreciated more as one of the most beautiful compositions in the scientific literature. As a conclusion to this article, we shall again quote from Sommerfeld, from his opening remark to Part III on the theory of relativity and electron theory: "The path taken by Einstein in 1905 in the discovery of the special theory of relativity was steep and difficult. ··· The path which we shall take is wide and effortless." The author wondered whether the path is as wide and effortless as the master said.

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